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LETTER TO THE EDITOR

General solution for a class of diffraction problems

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Abstract. Using a special mathematical technique previously developed by Malyuzhinets, the general solution of two-dimensional plane-wave diffraction problems in an angular region of any angle with boundary conditions containing spatial derivatives of arbitrary orders is constructed.

Mathematical theory of diffraction of waves by a wedge and similar structures such as a half-plane or a wedge-like hollow is widely used for studying a number of important problems in acoustics and electrodynamics connected with radiation and scattering from complicated targets, propagation over terrain relief and around artifical screens and barriers, etc. A broad spectrum of canonical problems for the simplest forms of boundary conditions (Dirichlet and Neumann) has been considered already at the beginning of this century in the classical works of Poincaré, Sommerfeld, Macdonald *et al* . It proves to be much more difficult to generalize these solutions to cases in which the boundaries of the region do not reflect perfectly and have absorbing or guiding properties. Methods based on the Wiener–Hopf technique (Noble 1958) are capable of giving elegant and efficient solutions to several important and interesting problems (for recent achievements see, for example, Rawlins 1976, Rawlins and Williams 1981, Abrahams and Wickham 1990). Unfortunately, they are fundamentally restricted to structures with rectangular geometries, such as half-planes and their junctions at right angles. A special mathematical technique has been developed by Malyuzhinets (1958a) to solve the canonical problem for an arbitrary-angle impedance wedge.

In the most general case, the properties of any plane surface Γ can be modelled by means of boundary conditions containing spatial high-order derivatives (Weinstein 1969)

$$\hat{A}\left(\frac{\partial}{\partial x}\right)\frac{\partial u}{\partial y} + \mathrm{i}k\hat{B}\left(\frac{\partial}{\partial x}\right)u|_{\Gamma} = 0 \tag{1}$$

where x and y are the coordinates, tangent and normal to the boundary, respectively, k denotes the wave number in the upper isotropic homogeneous half-space, $u(x, y)$ is a scalar function representing the wave field in the half space, $\hat{A}$, $\hat{B}$ are ordinary differential operators of even orders with constant coefficients. Using boundary conditions (1), one can describe, for example, electromagnetic scattering from bodies coated with dielectric materials (Weinstein 1969), anomalies of geomagnetic variations caused by underlying Earth's layers (Price 1949), radiation and diffraction of acoustic waves by thin elastic plates in fluids (Brechovskich 1973).

During the last three decades, starting from the work of Malyuzhinets (1958b) and Lamb (1959) on the diffraction of the sound by a thin elastic half-plane, considerable attention has

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been attracted to diffraction problems for wedge-like structures with high-order boundary conditions but all the solutions presented in the literature are concerned with either particular forms of the boundary conditions (Malyuzhinets and Tuzhilin 1970, Morgan and Karp 1974, Bernard 1987) or structures having rectangular geometry (Kouzov 1963, Belinskii *et al* 1973, Volakis and Senior 1987, Rojas *et al* 1991, Ricoy and Volakis 1992, Senior 1993).

The aim of this letter is to construct the general solution for two-dimensional plane wave diffraction problems in an angular region of any angle with boundary conditions taken in their general form (1). A technique is employed which has been previously proposed by Malyuzhinets (1958a,b,c). Mathematically, there is no essential difference between two-dimensional problems of acoustics and electrodynamics; therefore, a unified consideration of both cases is given below. A time dependence $e^{-i\omega t}$ is understood and supressed throughout.

Let us consider the diffraction of a plane wave $e^{-ikr\cos(\varphi-\varphi_0)}$ in an angular region $|\varphi| \leqslant \Phi$ $(0 < \Phi \leqslant \pi)$ with boundary conditions of arbitrary orders $N_\pm$ of type (1)

$$\hat{L}_\pm u|_{\pm\Phi} = 0 \tag{2}$$

where $\pm$ signs correspond to the upper ($\varphi = \Phi$) and lower ($\varphi = -\Phi$) faces of the region, respectively. In the polar coordinate system (r, φ), the boundary condition operators $\hat{L}_\pm$ can be expressed as

$$\hat{L}_\pm\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\varphi}\right) = \mp\hat{A}_\pm\left(\frac{\partial}{\partial r}\right)\frac{1}{r}\frac{\partial}{\partial\varphi} + ik\hat{B}_\pm\left(\frac{\partial}{\partial r}\right)$$

where $\hat{A}_\pm$ and $\hat{B}_\pm$ are assumed to be ordinary differential operators of arbitrary even orders with arbitrary constant coefficients. The total field $u(r, \varphi)$ must satisfy the Helmholtz equation, the edge conditions and the proper conditions at infinity.

According to Malyuzhinets' method, we represent the solution of the problem in the form of Sommerfeld's integral

$$u(r, \varphi) = \frac{1}{2\pi i}\int_\gamma e^{-ikr\cos\alpha} S(\alpha + \varphi)\, d\alpha$$

where the contour γ looks like two loops in the complex α plane (Malyuzhinets 1958a). To meet the conditions at the edge and at infinity, $S(\alpha)$ must be analytical inside the loops, have a simple pole with a unit residue at $\alpha = \varphi_0$, and be limited at the infinity point $\alpha = \infty$.

From boundary conditions (2), using Malyuzhinets' theorem (1958c), we obtain a system of functional equations

$$l_\pm(\alpha)S(\alpha \pm \Phi) - l_\pm(-\alpha)S(-\alpha \pm \Phi) = 2\sin\alpha \sum_{n=1}^{N_\pm} C_n^\pm \cos^{n-1}\alpha \tag{3}$$

which is completely equivalent to the boundary conditions and contains a set of arbitrary constants $C_n^\pm$. Variable coefficients $l_\pm(\alpha)$ in system (3) (i.e. symbols of the boundary condition operators),

$$l_\pm(\alpha) = L_\pm(-ik\cos\alpha, -ik\sin\alpha)$$

are polynomials of $\sin\alpha$. Zeros of these functions from the strip $|\operatorname{Re}\alpha| < \pi/2$

$$\alpha = \mp\theta_n^\pm \qquad n = 1, 2, \ldots, N_\pm$$

can be interpreted as complex Brewster angles of the wedge boundaries, i.e. the angles at which the Fresnel reflection coefficients vanish.

To solve system (3), it is convenient to introduce an auxiliary function $\Psi(\alpha)$, analytical in the strip $|\mathrm{Re}\,\alpha| \leqslant \Phi$, which is a partial solution of the corresponding homogeneous system (Malyuzhinets 1958a). This solution can be constructed via the Fourier transform and expressed as a combination of special Malyuzhinets functions $\psi_\Phi(\alpha)$

$$\Psi(\alpha) = \prod_{n=1}^{N_+} \left(\psi_\Phi\left(\alpha + \Phi + \frac{\pi}{2} - s_n^+\theta_n^+\right)\psi_\Phi\left(\alpha + \Phi - \frac{\pi}{2} + s_n^+\theta_n^+\right)\right)^{s_n^+}$$
$$\prod_{n=1}^{N_-} \left(\psi_\Phi\left(\alpha - \Phi + \frac{\pi}{2} - s_n^-\theta_n^-\right)\psi_\Phi\left(\alpha - \Phi - \frac{\pi}{2} + s_n^-\theta_n^-\right)\right)^{s_n^-}$$

where $s_n^\pm = \mathrm{sign}(\mathrm{Re}\,\theta_n^\pm)$, $n = 1, 2, \ldots, N_\pm$.

Then the substitution $S(\alpha) = \Psi(\alpha)\tilde{S}(\alpha)$ reduces system (3) to

$$\tilde{S}(\alpha \pm \Phi) - \tilde{S}(-\alpha \pm \Phi) = \frac{2 \sin\alpha \sum_{n=1}^{N_\pm} C_n^\pm \cos^{n-1}\alpha}{l_\pm(\alpha)\Psi(\alpha \pm \Phi)} \tag{4}$$

with constant coefficients.

The general solution of system (4) must consist of

(i) a meromorphic solution of the corresponding homogeneous system with the unit residue at a point $\alpha = \varphi_0$,

$$\sigma(\alpha, \varphi_0) = \frac{\mu \cos(\mu\varphi_0)}{\sin(\mu\alpha) - \sin(\mu\varphi_0)} \qquad \mu = \frac{\pi}{2\Phi}$$

well known from Sommerfeld's solutions for a perfectly reflecting wedge;

(ii) entire solutions of the homogeneous system (Tuzhilin 1970)

$$\sigma_N(\alpha) = \sum_{n=0}^{N} C_n \cos(\mu n(\alpha - \Phi))$$
$$\Omega_M(\alpha) = \sum_{m=0}^{M} \omega_m \cos(\mu m(\alpha - \Phi))$$

(iii) a partial solution of non-homogeneous system (4) (easily obtainable via Fourier transform)

$$\Lambda(\alpha) = \sum_{n=1}^{N_+} C_n^+ \Lambda_n^+(\alpha) + \sum_{n=1}^{N_-} C_n^- \Lambda_n^-(\alpha)$$

where

$$\Lambda_n^\pm(\alpha) = \mp\frac{\mu}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \tan\left(\frac{\mu}{2}(\alpha + \beta \pm \Phi)\right) \frac{\sin\beta \cos^{n-1}\beta \,\mathrm{d}\beta}{l_\pm(\beta)\Psi(\beta \pm \Phi)\sigma_N(\beta \pm \Phi)}.$$

Thus, the general solution of the initial system (3) which has the proper residue at $\alpha = \varphi_0$ can be written in the form

$$S(\alpha) = \Psi(\alpha)\left(\frac{\sigma(\alpha,\varphi_0)}{\Psi(\varphi_0)} + \sigma_N(\alpha)\Lambda(\alpha) + \Omega_{N-1}(\alpha)\right). \tag{5}$$

Note that the solution obtained contains a set of arbitrary constants $C_n^{\pm}$, C_n, ω_m. By matching them, we can ensure the belonging of $S(\alpha)$ to the class of analytical functions which has been described in the statement of the diffraction problem. Firstly, $S(\alpha)$ is a meromorphic function of α and all the poles of the function which can lead to a violation of the condition at $r = \infty$ must be forbidden. It gives a subset of conditions on the unknown coefficients

$$\frac{\sigma(\alpha_j^{\mp},\varphi_0)}{\Psi(\varphi_0)} + \sigma_N(\alpha_j^{\mp})\Lambda(\alpha_j^{\mp}) + \Omega_{N-1}(\alpha_j^{\mp}) = 0 \qquad j = 1, 2, \ldots \tag{6}$$

where $\alpha_j^{\mp}$ denote the forbidden poles.

Secondly, when $\operatorname{Im}\alpha \to \infty$, functions in (5) can be estimated as

$$\sigma(\alpha,\varphi_0), \Lambda(\alpha) = O(\exp(-\mu|\operatorname{Im}\alpha|))$$
$$\sigma_N(\alpha), \Omega_N(\alpha) = O(\exp(\mu N|\operatorname{Im}\alpha|))$$
$$\Psi(\alpha) = O\left(\exp\left(\frac{\mu}{2}(\lambda_+ + \lambda_-)|\operatorname{Im}\alpha|\right)\right) \qquad \lambda_{\pm} = \sum_{n=1}^{N_{\pm}} s_n^{\pm}.$$

So, if $\lambda_+ + \lambda_- \geqslant 3$, then solution (5) gets unbounded at a point $\alpha = \infty$. In this case, it is necessary to add to (6) a second subset of conditions following from conditions of cancellation for a few first terms in expansion of $S(\alpha)$ near $\alpha = \infty$:

$$\sum_{n=1}^{N_+} C_n^+ I_n^+(\mu m) + (-1)^m \sum_{n=1}^{N_-} C_n^- I_n^-(\mu m) = \frac{2}{\Psi(\varphi_0)} \sin\left(m(\mu\varphi_0 - \frac{\pi}{2})\right)$$
$$I_n^{\pm}(\mu m) = \mp\frac{1}{\pi}\int_{-i\infty}^{+ik\infty} e^{i\mu m\beta}\frac{\sin\beta\cos^{n-1}\beta}{l_{\pm}(\beta)\Psi(\beta\pm\Phi)}d\beta \tag{7}$$
$$m = 1, 2, \ldots, \left[\frac{\lambda_+ + \lambda_- - 1}{2}\right]$$

where square brackets mean the entire part of a quantity. Note that formulae (7) do not contain coefficients C_n and ω_m because in this case $\sigma_N(\alpha) = 1$ and $\Omega_{N-1}(\alpha) = 0$ to guarantee the boundedness of $S(\alpha)$.

Expression (5) completed by equations (6) and (7) describes the general solution of the initial diffraction problem, satisfying all the requirements (conditions at edge and $r = \infty$) which are usually imposed on solutions in diffraction theory. But the solution obtained is *not unique because the total number of equations (6) and (7) is still less then the number of* the unknown constants in representation (5). Similar facts are known to appear without fail when the order of boundary-condition operators exceeds that of the Helmholtz equation. A more detailed analysis shows that the number of undefined constants in the general solution is directly expressed through the orders of boundary condition operators

$$K = \tfrac{1}{2}(N_+ + N_- - 1)$$

and does not depend on the values of the parameters $\lambda_\pm$, i.e. on how the roots of functions $l_\pm(\alpha)$ are located on the complex α plane. This result coincides exactly with that obtained by Kouzov and his co-authers (Kouzov 1963, Belinskii *et al* 1973) for rectangular structures, using the Wiener–Hopf mathematical technique.

To extract a unique solution, it is necessary to impose a set of additional linearly independent conditions which describe in more detail the properties of the wave field near the irregularity point $r = 0$ where boundary conditions (2) containing spatial high-order derivatives are not applicable. There exist various ways of formulating these conditions and the first of them is to use the so-called 'boundary-contact conditions' that prescribe certain relations between the wave field and its derivatives at a singular point (Kouzov 1963):

$$\begin{aligned} &\hat{R}_n^+ u + \hat{R}_n^- u = 0 \qquad n = 1, 2, \ldots, K \\ &\hat{R}_n^\pm u = \lim_{r\to 0}\left(\mp \hat{H}_{1n}^\pm\left(\frac{\partial}{\partial r}\right)\frac{1}{r}\frac{\partial u}{\partial \varphi}|_{\pm\Phi} + \hat{H}_{2n}^\pm\left(\frac{\partial}{\partial r}\right) u|_{\pm\Phi}\right) \end{aligned} \tag{8}$$

where $\hat{H}_{mn}^\pm$, $m = 1, 2$, are polynomials. This approach is very useful in acoustics in solving problems of sound diffraction by corner junctions of thin elastic plates described classically ($N_\pm = 5, K = 4$) (Malyuzhinets 1958b, Lamb 1959, Kouzov 1963, Malyuzhinets and Tuzhilin 1970, Belinskii *et al* 1973) because the required first four derivatives,

$$\left(\frac{\partial}{\partial r}\right)^n \left(\frac{1}{r}\frac{\partial u}{\partial \varphi}|_{\pm\Phi}\right) \qquad n = 0, 1, 2, 3$$

have a clear physical meaning, expressing plate displacements, angles between vibrating plates, rotatory moments of plates, and forces acting on plates, respectively. Consequently, for this class of problems the boundary-contact conditions can be formulated in advance, corresponding to the kinematic and dynamic conditions that characterize the junction of the plates (free edges, rigidly fixed edges, hinge joint etc).

In electrodynamics, the use of conditions (8) seems to be not so convenient, due to the absence of any physical interpretation for quantities involved in the case of $K \geqslant 2$. To overcome these difficulties, a different type of the additional conditions has been proposed recently (Ricoy and Volakis 1992, Osipov 1992), based on certain relationships between the constants in the general solution and the coefficients in modal representations of the wave field.

Once the unknown constants have been determined, the solution obtained can be efficiently used in applied calculations, because the solution contains only well known special Malyuzhinets functions $\psi_\Phi(z)$ (see, for example, (Osipov 1990a)) and is expressed in terms of the Sommerfeld integral, which allows both analytical and numerical methods (Osipov 1990b,c, Osipov 1991).

Applications of the solution presented to various problems of acoustics and electrodynamics will be the subject of the forthcoming publications.

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